Chapter 0

0. Probability & Linear Algebra Review

1.1. Discrete Random Variables

Exercise 1:

Normalization constant for the Poisson distribution:

The probability mass function (pmf) of a **Poisson distribution** with parameter λ is given by:

$$(\lambda) = \frac{c\lambda^k}{k!}, k = 0, 1, 2, \ldots$$

Compute the value of *c*.

1) $e^{-\lambda}$

Exercise 2:

Moments of Bernoulli variables:

The *n*th **moment** of a random variable *X* is defined to be the expectation $E[X^n]$ of the n^{th} power of *X*.

Recall that a **Bernoulli random variable with parameter** *p* is a random variable that takes the value 1 with probability *p*, and the value 0 with probability 1−*p*.

Let *X* be a Bernoulli random variable with parameter 0.7. Compute the **expectation values** of X^k, denoted by $E[X^k]$, for the following three values of *k*: *k*=1,4, and 3203.

E[*X*]=? $E[X^4]$ =? $E[X^{3203}]$ =?

1) 0.7, 2) 0.7, 3) 0.7

Exercise 3:

Let X be a Bernoulli random variable with parameter $p \in [0,1]$. Compute the **variance** of X, which is denoted by $\mathrm{Var}[X]$.

$\mathrm{Var}[X]$=?

1) $p \cdot (1-p)$

What value(s) of the parameter p maximize the variance? What values minimize it?

(For each question, enter the values of p as a list of **numbers**, separated by commas. For example, to enter the set {0.2,0.3}, type **0.2, 0.3**, **without the braces**. Do NOT enter duplicate values, e.g. **0.2, 0.3, 0.3** will be graded as incorrect.)

The values of p for which $\mathrm{Var}[X]$ is minimized:

1) 0

The values of p for which $\mathrm{Var}[X]$ is maximized:

1) 0.5

Exercise 4:

Sum of Bernoulli variables:

Given n i.i.d. realizations $X_1,\ldots,X_n \sim \mathrm{Ber}(p)$, what is the distribution of $\Sigma_{i=1}^{n} X_i$

Binomial with parameters n and p

Exercise 5:

Discrete uniform random variables:

Recall that a **uniform random variable** is a random variable that takes values with equal probability,

Let X be a uniform random variable in the finite set {1, 2,…,20}.

Compute the following quantities.

The probability that X is an even number **P**(X is an even number)=?

The probability that X is a prime number: **P**(X is a prime number)=?

1) 0.5 2)0.4

Exercise 6:

Moments of Gaussian random variables:

Let X be a Gaussian random variable with mean μ and variance σ^2. Compute the following moments:

Remember that we use the terms **Gaussian random variable** and **normal random variable** interchangeably.

(Enter your answers in terms of μ and σ.)

$E[X^2]$=?

$E[X^3]$=?

$E[X^4]$=?

$Var[X^2]$=?

Write $\mathbf{P}(X>0)$ in terms of the **cumulative distribution function (cdf)** Φ of the standard Gaussian distribution, evaluated at a function of μ and σ.

Recall that: $\Phi(x) = P(Z \leq x), x \in R$

where $Z \sim N(0,1)$ is a standard normal variable. (Enter `Phi` for Φ, e.g. if the answer is $\Phi(\mu)$, enter `Phi(mu)`.)

1) $\mu^2 + \sigma^2$ 2) $\mu^3 + 3 \cdot \mu \cdot \sigma^2$ 3) $u^4 + 6 \cdot \mu^2 \cdot \sigma^2 + 3 \cdot \sigma^4$,

4) $2 \cdot \sigma^4 + 4 \cdot \mu^2 \cdot \sigma^2$, 5) $1 - \Phi(-\frac{\mu}{\sigma})$

Exercise 7:

Covariance of Gaussians:

Recall that **i.i.d.** stands for **independent and identically distributed** . A collection of random variables 1/2 are **i.i.d.** if all of them follow the same distribution, and each X_i does not contain information about the other realizations.

Let X, Y be i.i.d. **standard** normal random variables, that is, $X, Y \sim N(0,1)$.

Recall that the **covariance** of two random variables X and Y, denoted by $Cov(X,Y)$, is defined as:

$$Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$$

Compute the following variances and covariances:

$\mathrm{Var}(X+Y)$=?

$\mathrm{Var}(XY)$= ?

$\mathrm{Cov}(X,X+Y)$= ?

$\mathrm{Cov}(X,XY)$=?

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Exercise 8:

For each of the statements below, determine whether it is true (meaning, always true) or false (meaning, not always true).

1) For any two random variables, $\mathrm{Var}(X+Y)=\mathrm{Var}(X)+\mathrm{Var}(Y)$
2) If the covariance, $\mathrm{Cov}(X,Y)$ between two random variables X,Y is 0, then X and Y are independent.

False, False

Exercise 9:

Expectation, variance and probabilities:

Let X be a uniform random variable in the interval [2,8.5]. Find the following quantities (if needed, round to the nearest 10^{-4}):

$E[X]$=?

$\mathrm{Var}[X]$=?

$\mathbf{P}(X>4)$=?

$\mathbf{P}(\log(X)\leq 1)$= ?

5.25, 3.5208, 0.6923, 0.1105

Exercise 10:

Two independent copies:

Let U,V be i.i.d. random variables uniformly distributed in [0,1]. Compute the following quantities:

$E[|U-V|]$=?

$\mathbf{P}(U=V)$=?

P($U \leq V$)=?

0.33, 0, 0.5

Exercise 11:

Maximum of uniform random variables:

Let $U_1,\ldots,U_n$ be i.i.d. random variables uniformly distributed in [0,1] and let $M_n = \max_{1 \leq i \leq n} U_i$

Find the CDF of M_n, which we denote by $G(t)$, for $t \in [0,1]$.

For $t \in [0,1]$,

$G(t)$= ?

Now, let $F_n(t)$ denote the cdf of $n(1 - M_n)$; for $t > 0$

$\lim_{n \to \infty} F_n(t) =$?

$t^n, 1 - e^{-t}$

Exercise 12:

Sums and products:

Let X be an exponential random variable with parameter $\lambda > 0$ and Y be a Poisson random variable with parameter $\mu > 0$. Assume that X and Y are independent. Compute the following quantities:

$E[X^2+Y^2]$=?

$E[X^2Y]$=?

Var($2X+3Y$)=?

$\frac{2}{\lambda^2} + \mu + \mu^2, \frac{2 \cdot \mu}{\lambda^2}, \frac{4}{\lambda^2} + 9 \cdot \mu$

Exercise 13:

Estimators:

Let $X_1,\ldots,X_n$ be i.i.d exponential random variables with parameter λ and let $Z_i = \mathbf{1}(X_i \leq 1), i = 1,\ldots,n$. Recall that **1**($X \leq 1$) denotes the **indicator function** that takes the value 1 when $X \leq 1$ and 0 otherwise.

What is the limit in probability, as n goes to infinity, of $\frac{1}{n}\sum_{i=1}^{n} Z_i$?

$$\frac{1}{n}\sum_{i=1}^{n} Z_i \xrightarrow{P} n\to\infty$$

$$1 - e^{-\lambda}$$

Exercise 14:

Properties of the exponential distribution:

Let X be an exponential random variable with parameter λ=2 that models the lifetime (in years) of a lightbulb. Compute the probability that the lightbulb lasts for at least 2 years. Round your answer to the nearest 10^{-2}.

P($X \geq 2$)=?

Given the lightbulb has lasted 2 years, find the probability that it lasts for k more years for any positive integer k.

P($X \geq k+2 | X \geq 2$)=?

$$0.02, e^{-2 \cdot k}$$

Exercise 15:

Gaussian probabilities:

Let $X \sim N(1, 2.25)$ As a reminder, the 2.25 here represents the value of σ^2. Using the normal probability table below, compute the following probabilities:

P($X > 1$)=?

P($|X-2| \leq 1$)=?

P($X^2 > 4$)=?

P($X^2 - 2X - 1 > 0$)=?

0.5, 0.4087, 0.2753, 0.3472

Exercise 16:

Approximation of Binomial variables:

Using the normal probability table, evaluate approximately **P**(X>400), where X is a binomial random variable with parameters 1000 and .3.

P(X>400)≃ ?

0

Exercise 17:

Matrices and Vectors:

Matrix Multiplication :

Let **A**= $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & -4 \end{pmatrix}$ and let **B**= $\begin{pmatrix} -1 & 0 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. The dimensions of the product AB are: (rows and columns) ?

More generally, let **A** be an m×n matrix and **B** be an n×k matrix. What is the size of **AB**?

In addition, if **C** is a k×j matrix, what is the size of **ABC**?

2 rows and 3 columns, m rows and k column, m rows and j columns

Exercise 18:

Vector Inner product:

Suppose **u**= $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and **v**= $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ The product $\mathbf{u}^T\mathbf{v}$ evaluates the **inner product** (also called the **dot product**) of **u** and **v**, which evaluates to: ?

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Exercise 19:

Vector Outer product:

Suppose **u**= $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and **v**= $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ The product $\mathbf{u}^T\mathbf{v}$ evaluates the **outer product** which is a 2×2 matrix in this case:

What is $(\mathbf{u}\mathbf{v}^T)_{1,1}$?

What is $(\mathbf{u}\mathbf{v}^T)_{1,2}$?

What is $(\mathbf{u}\mathbf{v}^T)_{2,1}$?

What is $(\mathbf{u}\mathbf{v}^T)_{2,2}$?

-1, 1, -3, 3

Exercise 20:

Raw and Column Rank

Suppose **A**= $(1\ 3\ 2\ 6)$. The rows of the matrix, (1,3) and (2,6), span a subspace of dimension ? > This is the row rank of A.

The columns of the matrix $(1\ 2)and\ (3\ 6)$ span a subspace of dimension? This is the row rank of A.

1, 1

Exercise 21:

In general, row rank is always equal to the column rank, so we simply refer to this common value as the **rank** of a matrix.

What is the largest possible rank of a 2×2 matrix?

What is the largest possible rank of a 5×2 matrix?

In general, what is the largest possible rank of an *m*×*n* matrix?

2, 2, min(m,n)

Exercise 22:

What is the rank of:

$$(1\ 1\ 1\ 1)?$$

$$(1\ -1\ 1\ 0)?$$

$$(0\ 0\ 0\ 0)?$$

$$(1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0)?$$

$$(1\ 1\ 0\ 0\ -3\ 2\ 0\ 0\ 1)$$

1, 2, 0, 2, 3

This question is meant to serve as an answer to the following: If you sum two rank-1 matrices, do you get a rank-2 matrix? What about products? More generally, what rank is the sum of a rank-*r*1 and a rank-*r*2 matrix?"

Let

$$A = (-1\ 1\ -3\ 3), B = (1\ -1\ -1\ 1)and\ C = (0\ 0\ 0\ 1)and\ D = (1\ 1\ 1\ 1)$$

Observe that all four of these matrices are rank 1.

There are many ways to determine rank. Here is one useful fact that you could use for this problem:

“Every rank-1 matrix can be written as an outer product. Conversely, every outer product $\mathbf{uv}^T$ is a rank-1 matrix."

For example : $A = uv^T, B = vv^T, C = ww^T \text{ and } D = xx^T$

Where:

$$u = (1\ 3), v = (-\ 1\ 1), w = (0\ 1), x = (1\ 1)$$

Which combination of these matrices has rank 2? Choose all that apply.

A+C

Which combination of these matrices has rank 1? Choose all that apply.

A+A A+B AB AC

Exercise 23:

An $n \times n$ matrix **A** is invertible if and only if **A** has full rank, i.e. $\text{rank}(\mathbf{A}) = n$.

Which of the following matrices are invertible? Choose all that apply.

$$A = (1\ 3\ 2\ 6)$$

$$B = (1\ 2\ 2\ 1)$$

$$C = (1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1)$$

$$D = (2\ -\ 1\ -\ 1\ -\ 1\ 2\ -\ 1\ -\ 1\ -\ 1\ 2)$$

B AND C